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JONAH → Observation

MATH 307

OFFICE HOURS

TODAY 12:30 - 3:20
WED 12:00 - 2:00
FRI 12:30 - 3:30

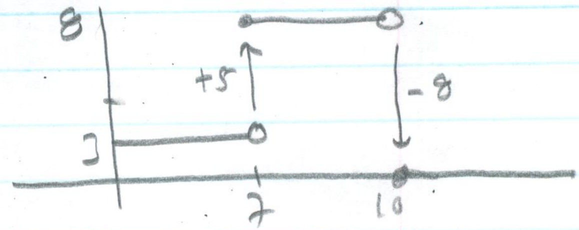
COMMUNICATION
[illegible]-006

HW 7, DUE FRIDAY

ENTRY TASK

$$f(t) = \begin{cases} 3, & 0 \le t < 7; \\ 8, & 7 \le t < 10; \\ 0, & 10 \le t. \end{cases}$$

Write $f(t)$ in terms of unit step functions.




$$f(t) = 3 + 5u_7(t) - 8u_{10}(t)$$

Recall: $u_c(t) = \begin{cases} 0, & t < c; \\ 1, & t \ge c. \end{cases}$

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t)\,dt$$

$$= \int_0^c e^{-st} \underbrace{u_c(t)}_{0} dt + \int_c^\infty e^{-st} \underbrace{u_c(t)}_{1} dt$$

$$= \lim_{A\to\infty} \int_c^A e^{-st}\,dt = \lim_{A\to\infty} \left. -\frac{1}{s} e^{-st} \right|_c^A$$

$$= \lim_{A\to\infty} \left[ \cancel{-\frac{1}{s} e^{-As}}^{\,0} + \frac{1}{s} e^{-cs} \right] \qquad s > 0$$

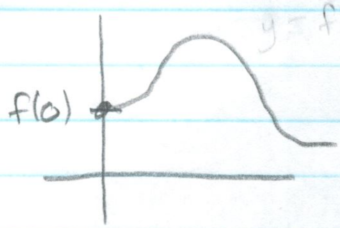
$$= \frac{e^{-cs}}{s}$$

$$\boxed{\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}}$$

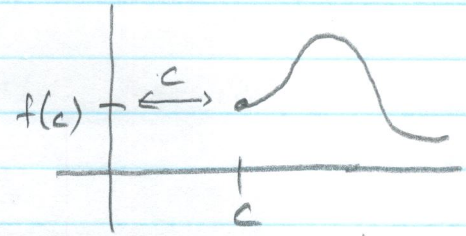
Ex) $\mathcal{L}\{3 + 5u_7(t) - 8u_{10}(t)\}$

$$= \frac{3}{s} + 5\frac{e^{-7s}}{7} - 8\frac{e^{-10s}}{10}$$

## SHIFTING

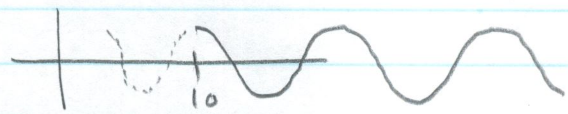



$y = f(t)$

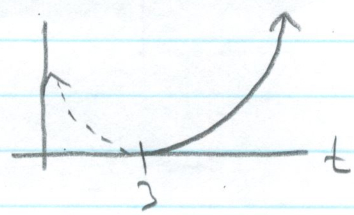



$y = f(t-c)$

Ex) $y = \cos(t-10)$

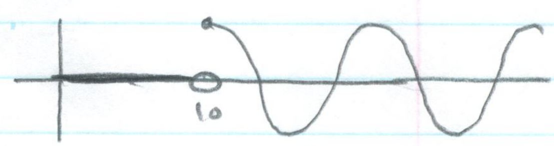



$y = (t-3)^2$

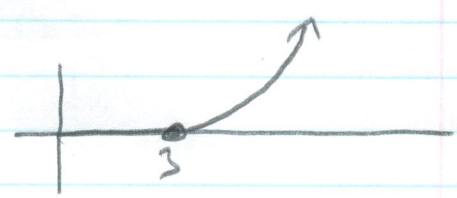



$$\boxed{u_c(t)\, f(t-c) = \begin{cases} 0, & t < c; \\ f(t-c), & t \geq c \end{cases}}$$

Ex) $y = u_{10}(t) \cos(t-10)$




$y = u_3(t)\,(t-3)^2$



$$\mathcal{L}\{u_c(t)f(t-c)\} = \int_0^\infty u_c(t) f(t-c) e^{-st}\,dt$$

$$= \int_c^\infty f(t-c)e^{-st}\,dt \qquad \begin{array}{l} u = t-c \\ du = dt \end{array}$$

$$= \int_0^\infty f(u) e^{-s(u+c)}\,du$$

$$= e^{-cs}\int_0^\infty f(u) e^{-su}\,du$$

$$= e^{-cs}\,\mathcal{L}\{f(u)\}$$

$\boxed{\text{Thm}}$ IF $F(s) = \mathcal{L}\{f(u)\}$

$\left(\text{so } \mathcal{L}^{-1}\{F(s)\} = f(u)\right)$

THEN

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(u)\} = e^{-cs}F(s)$$

$$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t-c)$$

$$= u_c(t)\,\mathcal{L}^{-1}\{F(s)\}(t-c)$$

$\boxed{\text{Ex}}$ $\mathcal{L}\{u_{10}(t)\cos(t-10)\} = e^{-10s}\,\mathcal{L}\{\cos(u)\}$

$$= e^{-10s}\frac{s}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{e^{-7s}\frac{s}{s^2+1}\right\} = u_7(t)\,\underbrace{\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}}_{\cos(t)}(t-7)$$

$$= u_7(t)\cos(t-7)$$

Ex) $f(t) = \begin{cases} 6, & 0 \le t < 3; \\ 6 + 2\sin(4(t-3)), & 3 \le t. \end{cases}$

$$f(t) = 6 + 2u_3(t)\sin(4(t-3))$$

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \mathcal{L}\{6\} + 2\,\mathcal{L}\{u_3(t)\sin(4(t-3))\} \\ &= \frac{6}{s} + 2e^{-3s}\,\mathcal{L}\{\sin(4u)\} \\ &= \frac{6}{s} + 2e^{-3s}\,\frac{4}{s^2+16} \\ &= \frac{6}{s} + \frac{8e^{-3s}}{s^2+16}\end{aligned}$$

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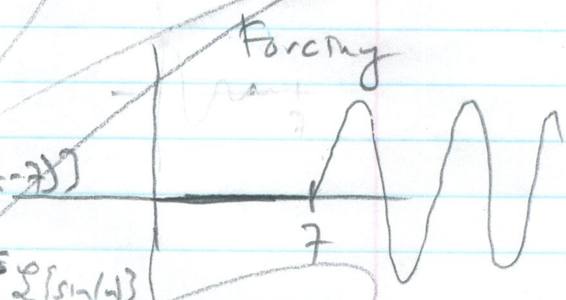
SEE OVERHEAD!

Preview of next lecture

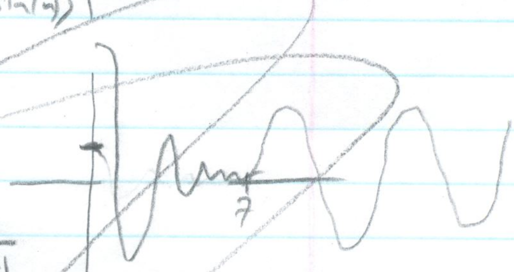
$$\underbrace{y'' + y' + 5y}_{\text{damped spring}} = \underbrace{\begin{cases} 0, & 0 \le t < 7; \\ 10\sin(t-7), & t \ge 7. \end{cases}}_{10u_7(t)\sin(t-7)} \qquad y(0)=20,\ y'(0)=0$$




$$\mathcal{L}\{y''\} + \mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = 10\,\mathcal{L}\{u_7(t)\sin(t-7)\}$$

$$s^2\mathcal{L}\{y\} - 20s + s\mathcal{L}\{y\} - 20 + 5\mathcal{L}\{y\} = 10e^{-7s}\,\mathcal{L}\{\sin(u)\}$$

$$(s^2+s+5)\mathcal{L}\{y\} - 20s - 20 = 10e^{-7s}\,\frac{1}{s^2+1}$$




$$\mathcal{L}\{y\} = 10e^{-7s}\,\frac{1}{(s^2+1)(s^2+s+5)} + \underbrace{\frac{20s+20}{(s^2+s+5)}}$$

$$\frac{10e^{-7s}}{17}\left[\frac{-s}{s^2+1} + \frac{4}{s^2+1} + \frac{(s+1)}{(s+1)^2+4} - \frac{4}{(s+1)^2+4}\right] + \frac{20(s+1)}{(s+1)^2+4}$$

$$y(t) = \frac{10}{17}u_7(t)\left[-\cos(t-7) + 4\sin(t-7) + e^{-(t-7)}\cos(2(t-7)) - 4\right. \quad + 20e^{-t}\cos(2t)$$